Stability Analysis of Degenerately-Damped Oscillations

Thomas Anderson[1], George Avalos[2], Elizabeth Galvin[2], Ian Kessler[3], Michelle Kleckner[4], Daniel Toundykov[2,⋆], William Tritch[5]

[1] Department of Computing & Mathematical Sciences, California Institute of Technology, CA 91125
[2] Department of Mathematics, University of Nebraska-Lincoln, NE 68588
[3] Carroll College, MT 59625
[4] School of Public Health, University of Michigan, MI 48104
[5] Department of Mathematics & Statistics, Texas Tech University, TX 79409

⋆ *Corresponding author:* dtoundykov@unl.edu

Abstract

Presented here is a study of well-posedness and asymptotic stability of a “degenerately damped” PDE modeling a vibrating elastic string. The coefficient of the damping may vanish at small amplitudes thus weakening the effect of the dissipation. It is shown that the resulting dynamical system has strictly monotonically decreasing energy and uniformly decaying lower-order norms, however, is *not uniformly stable* on the associated finite-energy space. These theoretical findings were motivated by numerical simulations of this model using a finite element scheme and successive approximations. A description of the numerical approach and sample plots of energy decay are supplied. In addition, for certain initial data the solution can be determined in closed form up to a dissipative nonlinear ordinary differential equation. Such solutions can be used to assess the accuracy of the numerical examples.

1 Introduction

Advances in nonlinear functional analysis and the rich theory of linear distributed parameter systems have led to a growing of body of work on nonlinear infinite-dimensional models. For instance, in a 2nd-order evolution framework (especially, wave, elastodynamics, or thin plates with no rotational inertia terms) for an appropriate elliptic operator A a linear equation with viscous damping for an unknown $u = u(t, x)$ may be expressed as

$$\ddot{u} + A(x)u + \beta(x)\dot{u} = F(t, x)$$

with $\beta > 0$. We will focus on the evolution on a bounded domain and under suitable homogeneous boundary conditions. A nonlinear refinement on the dissipative term may take the form of a feedback law $g(\dot{u})$. Stability properties of such models have been extensively analyzed. In an infinite-dimensional sitting such a nonlinear feedback may change the topology of the problem and uniform stability becomes reliant on the regularity of solutions (for example, see [1, 2, 3]).

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A more general scenario would account for coefficients that depend on the solution itself:

$$\ddot{u} + A(x)u + \beta(x, u, \dot{u})\dot{u} = F(t, x)\,.$$

Assuming the well-posedness of an associated initial-boundary value problem can be resolved, if the term $\beta(x, u, \dot{u})$ is not guaranteed to be strictly positive on a *fixed* appropriately configured set, then analysis of stability becomes much more involved since the region where the dissipation is active now evolves with the solution and may not always comply with the requirements of the geometric optics. The case when this coefficient vanishes at zero displacement, namely, $\beta(x, 0, \dot{u}) = 0$, will be referred to as *degenerate* damping.

Such a degeneracy naturally arises when investigating energy decay of higher-order norms. For example, the natural energy space for a semilinear wave problem

$$\ddot{u} - \Delta u + g(\dot{u}) = 0$$

is $u \in W^{1,2}(\Omega)$ and $\dot{u} \in L^2(\Omega)$. With more regular initial data one can consider behavior of higher-order energy norms, namely for $(u, u_t) \in W^{2,2}(\Omega) \times W^{1,2}(\Omega)$. One approach would be to differentiate the PDE in time which via the substitution $v = \dot{u}$ leads to a degenerately damped problem

$$\ddot{v} - \Delta v + g'(v)\dot{v} = 0$$

A particular example can be observed in the relation between Maxwell's system and the (vectorial) wave equation. For a given medium denote the electric permittivity by ϵ, magnetic permeability by λ and conductivity by σ. Then Maxwell's system reads

$$\begin{aligned} \dot{E} - \operatorname{curl}(\lambda H) + \sigma E &= 0 \\ \dot{H} + \operatorname{curl}(\epsilon E) &= 0\,, \end{aligned}$$

with $\operatorname{div}(\lambda H) \equiv 0$. On a bounded domain, subject to the *electric wall* boundary conditions, and for scalar-valued λ, ϵ, σ with positive lower-bounds, the term σE exponentially stabilizes this system [4]. In a more accurate nonlinear conduction model the coefficient σ may depend on the intensity of the electric field E. If we consider, for example, $\sigma = \alpha|E|^p$ for $p \geq 1$, then differentiating the first equation in time and combining with the equation for H gives

$$\ddot{E} + \operatorname{curl}(\lambda \operatorname{curl} \epsilon E) + \alpha p|E|^{p-2}E \cdot \dot{E}E + \alpha|E|^p\dot{E} = 0\,.$$

For example, taking, $p = 1$ gives

$$\ddot{E} + \operatorname{curl}(\lambda \operatorname{curl} \epsilon E) + \alpha(E \cdot \dot{E})\hat{E} + \alpha|E|\dot{E} = 0$$

where $\hat{E}$ is the normalized vector E. The term $\alpha|E|\dot{E}$ has features of the viscous dissipation in this second-order equation, but nonlinear conductivity augments it with a degenerate coefficient $\alpha|E|$.

The study of stability for the above models is much more delicate than in the situations where the damping, even nonlinear, depends on the time-derivative only. Weighted energy methods—from basic energy laws to Carleman estimates (e.g. [5, 6, 7, 8, 9, 10, 11])—have been successfully used to derive stabilization and observability inequalities for distributed parameter systems. However, these methods typically rely on the properties of the coefficients to ensure that suitable geometric

optics conditions are satisfied and the control effect suitably "propagated" [12] across the physical domain. One can sometimes dispense with geometric optics requirements for smooth enough initial data [13, 14, 15, 16, 17], yet even then the support of the control/damping term must contain a subset that is time-invariant (and with any time-dependent coefficients being non-vanishing, e.g., as in [15]). In turn, the analysis of control-effect propagation when the coefficients themselves depend on the solution and possibly go to zero wherever and whenever the solution does would require new techniques.

1.1 The model

The following semilinear model, if recast in a higher-dimensional setting becomes highly non-trivial even when just regarding local wellposedness. In a one-dimensional framework the nonlinearity is more tractable, but the rigorous stability analysis has long been open. We focus on an elastic string with a *degenerate damping*, namely a dissipative term whose coefficient depends on and may vanish with the amplitudes:

$$\ddot{u} - u_{xx} + f(u)\dot{u} = 0, \quad \text{for} \quad x \in \Omega := (0,1), \quad t > 0 \tag{1}$$

fixed at the end-points

$$u(t,0) \equiv u(t,1) \equiv 0 \quad \text{for} \quad t \geq 0 \tag{2}$$

and with a prescribed initial configuration at $t = 0$:

$$u(0,x) = u_0(x), \quad \dot{u}(0,x) = u_1(x) \quad \text{for a.e.} \quad x \in (0,1)\,. \tag{3}$$

The initial data u_0, u_1 live in the natural function spaces revisited below. Function f is assumed to be continuous non-negative, hence the term $f(u)\dot{u}$ a priori should provide some form of energy dissipation in the model.

The scenario of interest is when $f(s) \to 0$ as $s \to 0$, essentially causing the dissipative effect to deteriorate at small amplitudes. We will focus on the polynomial case

$$f(s) := \alpha s^{2m}, \quad \alpha > 0, \quad m \in \mathbb{N}\,. \tag{4}$$

satisfying the locally Lipschitz estimate

$$f(s) - f(r) = M(s,r)\cdot(s-r) \quad \text{with} \quad M(s,r) = \alpha \sum_{j=0}^{2m-1} s^j r^{2m-1-j}\,. \tag{5}$$

1.2 Known results and new challenges

Existence and uniqueness of weak *finite-energy* solutions to (1)–(3) was proven in [18] by means of Galerkin approximations. The advantage of a 1D framework is that the displacement function is absolutely continuous, hence topologically $f(u)\dot{u}$ is still in $L^2(0,1)$, as in the case of the corresponding linear model. However, in higher-dimensional analogues this embedding property is lost and proving existence becomes a markedly more complex task. First, fractional damping exponents were considered in order to ensure that the damping term is bounded with respect to the finite energy topology [19]. Arbitrary damping exponents were subsequently examined in [20, 21]. Due to

the loss of regularity solutions had to be characterized via a variational *inequality* and established by a rather technical application of Kakutani's fixed point theorem.

On the other hand, stability analysis even in one dimension poses a challenge that has been open for a number of years. Despite the gain in regularity, attributed to Sobolev embeddings, the key difficulty now is that energy estimates require some sort of information on the region where the damping is supported. In (1) both the magnitude and the support of the damping coefficient evolve with the geometry of the state, rendering all standard techniques inapplicable.

It is plausible to assume that some sort of a logarithmic uniform decay rate can be verified, possibly by combining ODE techniques (e.g. [22, 23]) with pointwise Carleman-type estimates. Another, though a rather weak, tentative indication of this outcome would be the uniform stability of the corresponding finite-dimensional analogue (see the Appendix). Yet the situation in infinite dimensions turns out rather different.

1.3 Contribution of this work

The goal of this article is to examine analytically and numerically stability properties of the dynamical system associated to (1)–(3):

- Establish global persistence of regularity in solutions with smooth initial data. Besides theoretical interest such a result is useful to justify the convergence estimates for numerical approximations.

- Prove that a polynomial degeneracy in the damping of the form (4) yields a system that is **not uniformly stable**.

- Present a numerical scheme that indicates the loss in decay rates. Such observations had been performed first and, in fact, served as a motivation for the theoretical results presented here.

1.4 Outline

The notation employed throughout the paper is summarized in Section 2.1. The two main results on well-posedness and stability are stated in Section 3.2.

Several auxiliary technical definitions used in the proofs can be found in Section 3.1. Local and global wellposedness are verified respectively in Sections 4.4 and 4.5. They draw upon two regularity lemmas proved earlier in Section 4.3.

The proof of the lack of uniform stability is the subject of Section 5. Numerical results are the subject of Section 6.

The Appendix contains results pertaining to the ODE analog of the considered problem, namely, a damped harmonic oscillator with the damping coefficient dependent on the displacement.

2 Preliminaries

2.1 Notation

This section serves as a quick reference for the basic notation used thorough the paper with some of the symbols revisited and discussed in more detail later in the text.

Henceforth $\|\cdot\|_X$ will denote the norm on a normed space X. For the space $L^2(\Omega)$ we will use

$$|u|_0 := \|u\|_{L^2(\Omega)},$$

with the corresponding inner product denoted by $(\cdot,\cdot)_0$. We will also frequently involve the Sobolev space

$$H_0^1(\Omega) := W_0^{1,2}(\Omega)$$

associated to an equivalent inner product and norm

$$(u,v)_1 = (u_x, v_x)_0 \quad |u|_1 := \sqrt{(u,u)_1}\,.$$

The bilinear form $\langle\cdot,\,\cdot\rangle$ will indicate the pairing of $H_0^1(\Omega)$ and its continuous dual $H^{-1}(\Omega)$.

We will also frequently use spaces of the form

$$C^n([0,T];X) \quad \text{or} \quad L^p(0,T;X),$$

which will be abbreviated respectively as

$$C_T^n X \quad \text{and} \quad L_T^p X\,.$$

Looking ahead, for the one-dimensional Dirichlet Laplacian operator A (discussed below) let us introduce the space

$$S_T^n := \left\{z \;\mid\; z \in C_T^j(\mathscr{D}(A^{\frac{n+1-j}{2}})) \quad \text{for} \quad j = 0,1,\ldots,n+1)\right\} \tag{6}$$

equipped with the natural graph norm. For example, $S_T^0 = C([0,T];\mathscr{D}(A^{1/2})) \cap C^1([0,T];L^2(\Omega))$ indicates the standard regularity in space and time for a weak solution to a linear wave equation. In turn $S_T^1 = C_T\mathscr{D}(A) \cap C_T^1\mathscr{D}(A^{1/2}) \cap C_T^2 L^2(\Omega)$ does the same for a strong solution to such an equation.

We will also be using spaces

$$\mathscr{H}^n := \mathscr{D}(A^{\frac{n+1}{2}}) \times \mathscr{D}(A^{\frac{n}{2}})\,.$$

Thus $\mathscr{H}^0$ is the natural finite energy state space for a linear wave problem and $\mathscr{H}^1$ denotes the domain of the corresponding evolution generator.

Relation $a \lesssim b$ will occasionally be used to indicate that $a \le Cb$ for a constant C which only depends on the main parameters of the system, e.g., the size of the domain Ω or the exponent p of the coefficient in (4).

2.2 Laplace operator

For convenience let us summarize some of the fundamentals. Consider the operator

$$A = -\partial_{xx} : \mathscr{D}(A) \subset L^2(\Omega) \to L^2(\Omega), \quad \mathscr{D}(A) = W^{2,2}(\Omega) \cap H_0^1(\Omega) \tag{7}$$

which is positive, self-adjoint with compact resolvent, and has eigenvalues

$$\lambda_n = n^2\pi^2, \quad n \in \mathbb{N}$$

with the corresponding eigenfunctions

$$E_n(x) = \sqrt{2}\sin(n\pi x), \quad n \in \mathbb{N}\,.$$

The E_n's form an orthonormal basis for $L^2(\Omega)$. For $r \in \mathbb{R}$ we can define fractional powers of A:

$$A^r\left(\sum_{n=1}^{\infty} c_n E_n\right) = \sum_{n=1}^{\infty} \lambda_n^r c_n E_n$$

with

$$\mathscr{D}(A^r) = \left\{\sum_{n=1}^{\infty} c_n E_n \quad : \quad \sum_{n=1}^{\infty} |\lambda_n|^{2r} |c_n|^2 < \infty\right\}.$$

The eigenfunctions $\{E_n\}$ form an orthogonal basis for every $\mathscr{D}(A^r)$, $r \in \mathbb{R}$. Some of the fractional powers can be identified with Sobolev spaces, e.g.

$$\mathscr{D}(A^{1/2}) = H_0^1(\Omega) \quad \text{and} \quad \mathscr{D}(A^{-1/2}) = [\mathscr{D}(A^{1/2})]' = H^{-1}(\Omega)\,.$$

Since in our situation the model is one-dimensional, then trivially no issues in these identifications arise in regard to the regularity of the domain. Operator A also corresponds to the Riesz isomorphism $H_0^1(\Omega) \to H^{-1}(\Omega)$, and for $u, v \in H_0^1(\Omega)$ we have

$$\langle Au, v\rangle = (u, v)_{1},.$$

2.3 State space and linear group generator

The natural finite-energy state space associated with the evolution driven by (1)–(2) is

$$\mathscr{H}^0 := \mathscr{D}(A^{1/2}) \times L^2(\Omega)\,.$$

If we set $y = (u, \dot{u})$ we can recast this problem as an evolution equation

$$y' = \mathbb{A}y$$

for the skew-adjoint operator $\mathbb{A} : \mathscr{D}(\mathbb{A}) \subset \mathscr{H}^0 \to \mathscr{H}^0$

$$\mathbb{A} = \begin{bmatrix} 0 & I \\ -A & 0 \end{bmatrix}$$

with

$$\mathscr{D}(\mathbb{A}) = \mathscr{D}(A) \times \mathscr{D}(A^{1/2})\,.$$

We will also consider smoother solutions for which we define

$$\mathscr{H}^n := \mathscr{D}(\mathbb{A}^n) = \mathscr{D}(A^{\frac{1+n}{2}}) \times \mathscr{D}(A^{n/2}) \quad \text{for} \quad n \in \mathbb{N} \tag{8}$$

with the associated graph norm given via

$$\begin{aligned} \|(u_0, u_1)\|^2_{\mathscr{H}^n} &= \|u_0\|^2_{\mathscr{D}(A^{(n+1)/2})} + \|u_1\|^2_{\mathscr{D}(A^{n/2})} \\ &= \sum_{k=1}^{\infty} \lambda_k^n \left(\lambda_k |\mathcal{F}_k[u_0]|^2 + |\mathcal{F}_k[u_1]|^2\right)\,. \end{aligned} \tag{9}$$

Here $\mathcal{F}_k$ denotes the k-th Fourier coefficient with respect to the Hilbert basis of $L^2(\Omega)$ given by the eigenfunctions E_k of A. Note also that in one-dimension the following continuous injection holds

$$\mathscr{H}^n \subset H^{n+1}(\Omega) \times H^n(\Omega) \hookrightarrow C^n(\overline{\Omega}) \times C^{n-1}(\overline{\Omega}) \tag{10}$$

for $n \in \mathbb{N} \cup \{0\}$ if we adopt the notation convention $C^{-1}(\overline{\Omega}) := L^2(\Omega)$.

3 Results

We start with a formalized notion of a solution to the class of PDE systems of the form (1)–(3). The following slightly more abstract formulation will help streamline the subsequent discussion.

3.1 Auxiliary definitions

Definition 3.1 (Wave problem)**.** *Let* $[\![F, \phi_0, \phi_1]\!]$ *be the shorthand for the initial-boundary value problem:*

$$\ddot{\phi} - \phi_{xx} = F, \quad \textit{for} \quad x \in \Omega,\ t > 0\,,$$

with the indicated derivatives taken in the sense of distributions, and subject to boundary conditions

$$\phi(t, 0) \equiv 0 \equiv \phi(t, 1) \quad \textit{for} \quad t > 0$$

and initial data

$$\phi(0, x) = \phi_0(x), \quad \dot{\phi}(0, x) = \phi_1(x) \quad \textit{for a.e.} \quad x \in \Omega\,.$$

Definition 3.2 (Weak solution of linear problem)**.** *Suppose* $(u_0, u_1) \in \mathscr{H}^0$ *and for some* $T > 0$, $F \in L^1_T L^2(\Omega)$. *Then we say a function*

$$u \in C^1_T L^2(\Omega) \cap C_T H^1_0(\Omega) \quad \big(\textit{equivalently } u \in S^0_T \textit{ or } (u, \dot{u}) \in C_T \mathscr{H}^0\big) \tag{11}$$

is a weak solution to $[\![F, u_0, u_1]\!]$ *on interval* $[0, T]$ *if*

(i) $u(0, x) = u_0(x)$ *and* $\dot{u}(0, x) = u_1(x)$ *for a.e.* $x \in \Omega$,

(ii) For any $\phi \in L^2(\Omega)$ *the scalar map* $t \mapsto (\dot{u}(t), \phi)_0$ *is absolutely continuous (hence a.e. differentiable) on* $[0, T]$.

(iii) For any $\phi \in H^1_0(\Omega)$

$$\frac{d}{dt}(\dot{u}(t), \phi)_0 + (u(t), \phi)_1 = \big(F(t), \phi\big)_0 \quad \textit{for a.e.} \quad t \in (0, T)\,. \tag{12}$$

Definition 3.3 (“Regular” functions)**.** *A function* u *on* $[0, T] \times \Omega$ *will be described as* ***regular of order*** $n \in \mathbb{N} \cup \{0\}$ *if it is continuously differentiable in time with the following regularity*

$$(u, \dot{u}) \in C_T \mathscr{H}^n\,. \tag{13}$$

In classical terminology, weak solutions correspond to order 0 *and strong solutions to order* 1.

Suppose $u = u(t,x)$ has the weak regularity (11) (regular of order 0). Then according to the (1D) Sobolev embedding $W^{m,2}(\Omega) \hookrightarrow C^{m-1}(\overline{\Omega})$ for $m \in \mathbb{N}$, the function $F(t,x) = f(u(t,x))\dot{u}(t,x)$ is well-defined as an element of $L^2_T L^2(\Omega)$. In fact we will generalize this statement for the purposes of subsequently analyzing more regular solutions.

Proposition 3.1. *Let $f(s) = \alpha s^{2m}$ for $m \in \mathbb{N}$. If $z \in S^n_T$, then*

$$A^{s/2}[f(z)\dot{z}] \in C^j_T \mathscr{D}(A^{\frac{n-s-j}{2}}), \quad \text{for} \quad s, j \in \mathbb{N} \cup \{0\}, \quad s+j \le n\,. \tag{14}$$

In addition,

$$\|\partial_t^j (f(z)\dot{z})\|_{C_T L^2(\Omega)} \le \mathcal{P}\left[\|\partial_t^k z\|_{C_T H^1_0(\Omega)}\right]_{k=1,\dots,j-1} (1 + \|\partial_t^j \dot{z}\|_{C_T L^2(\Omega)}), \quad j \le n, \tag{15}$$

where $\mathcal{P}$ is a polynomial in $j-1$ variables.

Example 3.1. *Due to a variety of spaces and indices involved in the statement of Proposition 3.1, it is helpful to look at a basic example. Take $\alpha = m = 1$, so $f(z)\dot{z} = z^2\dot{z}$ and consider the regularity order $n = 4$. Then the condition $z \in S^n_T$ reads*

$$z \in \bigcap_{j=0}^{5} C^j_T(\mathscr{D}(A^{\frac{5-j}{2}})) \tag{16}$$

In particular, $z \in C_T\mathscr{D}(A^{5/2})$, which corresponds to 5 square-integrable derivatives, first three of which satisfy zero boundary conditions. We have, for example,

$$\partial_t^4[f(z)\dot{z}] \;=\; 30\dot{z}\ddot{z}^2 + 20\dot{z}^2\dddot{z} + 20z\,\ddot{z}\,\dddot{z} + 10z\dot{z}\partial_t^4 z + z^2\boxed{\partial_t^4\dot{z}}\,.$$

Thus, for instance, $|\partial_t^4 f(z)\dot{z}|_0$ can be estimated using a polynomial of $L^\infty(\Omega)$ bounds on the functions z, $\dot{z}$, $\dots$, $\partial_t^4 z$, and one term involving the $L^2(\Omega)$ norm of the fifth derivative of z in time or, equivalently, $|\partial_t^4\dot{z}|_0$. This is precisely the conclusion of (15).

Likewise, if we consider, say, the 2-nd derivative in space and 2-nd in time to $s = 2$, $j = 2$ in (14) we get:

$$\partial_t^2[f(z)\dot{z}] = 2\dot{z}^3 + 6z\dot{z}\ddot{z} + z^2\dddot{z} \tag{17}$$

$$\begin{aligned}\partial_x^2\partial_t^2[f(z)\dot{z}] =& 12\dot{z}_x^2\dot{z} + 12z_x\ddot{z}\dot{z}_x + 12z_x\dot{z}\ddot{z}_x + 2z_x^2\dddot{z} + 4z\dddot{z}_x z_x + 12z\ddot{z}_x\dot{z}_x + 6\dot{z}z_{xx}\ddot{z} + 6\dot{z}^2\dot{z}_{xx} \\ &+ 2z\dddot{z}z_{xx} + 6z\ddot{z}\dot{z}_{xx} + 6z\dot{z}\ddot{z}_{xx} + z^2\boxed{\dddot{z}_{xx}}\end{aligned} \tag{18}$$

We have that $\partial_t^2[f(z)\dot{z}]$ has zero trace as follows from (17) and the zero boundary condition on z. Moreover, the highest-order term $\dddot{z}_{xx}$ in $\partial_x^2\partial_t^2[f(z)\dot{z}]$ can be bounded in $L^2(\Omega)$ since by (16) we have $\dddot{z} \in \mathscr{D}(A)$. The rest of the terms are in fact in $L^\infty(\Omega)$. This confirms that $A\partial_t^2[f(z)\dot{z}] \in C_T L^2(\Omega)$ in agreement with (14).

Note that if we had, say, $n = 6$ then in the same context we would need to prove that $A\partial_t^2[f(z)\dot{z}] \in C_T\mathscr{D}(A)$ instead of just $C_T L^2(\Omega)$. First of all, $n = 6$ would give $z \in C_T\mathscr{D}(A^{7/2})$ implying that $z, z_x, \dots, \partial_x^5 z$ have zero traces. Hence so do their time-derivatives and then it is immediately follows that $A\partial_t^2[f(z)\dot{z}]$ satisfies zero trace condition. And taking two more derivatives in space in (18) yields the highest-order term $\partial_x^4\partial_t^3 z$ which is in $L^2(\Omega)$ from $z \in S^6_T$ that implies $\dddot{z} \in C_T\mathscr{D}(A^2)$. Thus, $\partial_x^4\partial_t^3 z \in L^2(\Omega)$ and whereas other summands in $\partial_x^2 A\partial_t^2[f(z)\dot{z}]$ belong to $L^\infty(\Omega)$. We conclude that if $n = 6$, then $A\partial_t^2[f(z)\dot{z}]$ (is in $L^2(\Omega)$) has zero trace and integrable second derivative, again in accordance with (14).

Proof. Let

$$F(z) := f(z)\dot{z}$$

and first let's show that $\partial_t^j F$ belongs to $C_T\mathscr{D}(A^{(n-j)/2})$. For $j \leq n$, $\partial_t^j F$ is a polynomial in the following variables

$$\partial_t^j F = \mathcal{P}[f^{(k)}(z),\ \partial_t^k z,\ \partial_t^k \dot{z}]_{k=0,1,\dots,j}$$

that is affine with respect to the highest-order derivative $\partial_t^j \dot{z} = \partial_t^{j+1} z$, which is at least in $C_T L^2(\Omega)$ by the assertion that $z \in S_T^n$ (recall (6) and plug $j = n+1$). We can just bound coefficients of $\partial_t^{j+1} z$ using the fact that $H_0^1(\Omega)$ embeds continuously into $C(\overline{\Omega})$. Thus the bound (15) follows.

Next, $\mathscr{D}(A^{\frac{n+1-j}{2}})$ embeds into $C^{n-j}(\overline{\Omega})$ for $j \leq n$, so all of the terms in $\partial_x^{n-j}\partial_t^j F$ are $C_T C(\overline{\Omega})$ except for the highest order term $\partial_x^{n-j}\partial_t^j \dot{z} \in C_T L^2(\Omega)$. Since $\partial_x^{n-j}\partial_t^j F$ is affine with respect to that term with continuous coefficients, then

$$\partial_x^{n-j}\partial_t^j F \in C_T L^2(\Omega)\,.$$

It follows that

$$\partial_t^j F \in C_T W^{n-j,2}(\Omega) \quad \text{for} \quad j \leq n\,.$$

To strengthen this regularity to $C_T\mathscr{D}(A^{\frac{n-j}{2}})$ we must verify the boundary conditions. Since $\mathscr{D}(A)$ coincides with the $W^{2,2}(\Omega)$ functions that have zero trace, then it is sufficient to show that

$$\partial_x^k \partial_j F = 0 \quad \text{on} \quad \partial\Omega \quad \text{for} \quad 0 \leq k+j \leq n-2 \quad (k, j \geq 0) \tag{19}$$

That is, we can show that every derivative of total order (time + space) up to $n-2$ of F vanishes on the boundary. The asserted regularity $z \in S_T^n$ implies that $\partial_x^j \partial_t^k z$ has zero trace for any $j \leq n-1$ and any k. Since any $(n-2)$-order (space and time together) derivative of F involves at most $(n-1)$-st order terms in z, then (19) readily follows. Thus

$$\partial_t^j F \in C_T\mathscr{D}(A^{\frac{n-j}{2}})\,.$$

Because $A^{s/2}$ is by definition a bounded operator on $\mathscr{D}(A^{s/2})$, then for $\frac{s}{2} \leq \frac{n-j}{2}$ we have

$$A^{s/2}\partial_t^j F = \partial_t^j A^{s/2} F \in C_T\mathscr{D}(A^{\frac{n-j-s}{2}})\,.$$

confirming (14). □

Relying on (a special case of) Proposition 3.1 we formulate the notion of solution to (1)–(3).

Definition 3.4 (Weak solution to (1)–(3)). *We say u is a weak solution to* (1)–(3) *on* $[0,T]$ *if it is a weak solution to* $[\![F, u_0, u_1]\!]$ *with* $F(t,x) = f(u(t,x))\dot{u}(t,x)$ *(which is in* $L_T^2 L^2(\Omega)$ *by Proposition 3.1 for* $n = s = j = 0$*).*

Definition 3.5 (Energy). *For a function $z = z(t,x)$ define the quadratic energy functional of order n by*

$$E_z^{(n)}(t) := \frac{1}{2}|\partial_t^n z(t)|_1^2 + \frac{1}{2}|\partial_t^n \dot{z}(t)|_0^2 \quad \textit{with} \quad E_z := E_z^{(0)}\,. \tag{20}$$

3.2 Main theorems

With the above definitions in mind, the new results of interest are:

Theorem 3.1 (**Global well-posedness of weak and regular solutions**)**.** *Let f be as in* (4) *with exponent $2m$, $m \in \mathbb{N}$. Suppose for some integer $n \geq 0$ the initial condition* (3) *satisfies*

$$(u_0, u_1) \in \mathscr{H}^n := \mathscr{D}(A^{(n+1)/2}) \times \mathscr{D}(A^{n/2}) .$$

Then there exists a unique weak solution u, regular of order n (Definition 3.3), of (1)–(3) *on* $[0, T]$ *for any $T > 0$. Moreover $u \in S^n_T$, that is, $u \in C^j_T \mathscr{D}(A^{\frac{n+1-j}{2}})$ for $j = 0, 1, \ldots, n$.*

Example 3.2. *If a solution to* (1)–(3) *with $f(u) = u^2 \dot{u}$ has initial data given by the eigenfunctions of A (every $\mathscr{D}(A^r)$), then $u \in C^\infty([0, \infty); W^{p,\infty}(\Omega))$ for every $p \geq 1$.*

Theorem 3.2 (**Lack of uniform stability**)**.** *The dynamical system generated by* (1)–(3) *on the state space corresponding to weak solutions $\mathscr{H}^0 := H^1_0(\Omega) \times L^2(\Omega)$ is non-accretive, but is **not** uniformly stable. Specifically, the energy functional $t \mapsto E_u(t)$ is continuous non-increasing; however, for any constants $0 < r < \overline{r}$ and any time $\overline{T} > 0$ there exists an initial datum $(\overline{u}_0, \overline{u}_1) \in B_{\overline{r}}(0)$ such that the corresponding solution trajectory on $[0, \overline{T}]$ does not intersect $B_r(0)$.*

However, for any $s \in (0, 1)$ the lower-order norm $\|u(t)\|_{W^{s,2}(\Omega)}$ decays to zero as $t \to \infty$ with the decay rate uniform with respect to bounded sets of initial data. In addition, $E_u(t)$ is strictly monotonically decreasing on any interval where $E_u(t)$ is positive.

Remark 3.1 (Open problem)**.** *The question of* strong *stability of the dynamical system generated by* (1)–(3) *on the finite energy space $\mathscr{H}^0$ remains open, as far as we are aware. That is, for any given solution can we prove that $\lim_{t\to\infty} E_u(t) = 0$?*

4 Well-posedness

4.1 Linear problem

For convenience let us summarize a few classical results that can be easily verified, for example, using separation of variables.

Lemma 4.1. *Consider the problem $[\![F \equiv 0, u_0, u_1]\!]$ with $(u_0, u_1) \in \mathscr{H}^0$. Then there exist a group $t \mapsto \mathcal{S}(t)$ of linear operators on $\mathscr{H}^0$ such that $(u(t), \dot{u}(t)) := \mathcal{S}(t)(u_0, u_1)$ determines a weak solution to this problem on every $[0, T]$, $T > 0$. The group is explicitly given by*

$$\mathcal{S}(t)(u_0, u_1) := \sum_{k=1}^{\infty} \begin{bmatrix} \cos(\sqrt{\lambda_k} t) & \frac{1}{\sqrt{\lambda_k}} \sin(\sqrt{\lambda_k} t) \\ -\sqrt{\lambda_k} \sin(\sqrt{\lambda_k} t) & \cos(\sqrt{\lambda_k} t) \end{bmatrix} \begin{bmatrix} \mathcal{F}_k[u_0] \\ \mathcal{F}_k[u_1] \end{bmatrix} : \begin{bmatrix} E_k \\ E_k \end{bmatrix} \tag{21}$$

where $\{(\lambda_k, E_k)\}$ are the eigenvalue-eigenfunction pairs for the operator A defined in (7)*, and $\mathcal{F}_k$ is the k-th Fourier coefficient with respect to the Hilbert basis $\{E_k\}$ for $L^2(\Omega)$:*

$$E_k(x) = \sqrt{2} \sin(k\pi x) \quad \textit{and} \quad \lambda_k = k^2 \pi^2 .$$

Moreover, $\mathcal{S}(t)$ is a unitary operator on the space $\mathscr{D}(\mathbb{A}^k)$ with respect to the norm given by (9)*, as follows by direct calculation using* (21)*.*

The next result is likewise well-known:

Proposition 4.1 (Inhomogeneous linear problem)**.** *Consider the problem* $[\![F, w_0, w_1]\!]$ *with*

$$(w_0, w_1) \in \mathscr{H}^0 \,. \tag{22}$$

If $F \in L^2_T L^2(\Omega)$*, this problem possesses a unique weak solution* $w \in S^0_T$*. Then the continuous mapping* $t \mapsto E_w(t)$ *for energy functional* (20) *satisfies*

$$E_w(t) + \int_0^t (F(s), \dot{w}(s))_0 \, ds = E_w(0) \quad \textit{for all} \quad t \in [0, T] \,. \tag{23}$$

In particular, from the Gronwall estimate it readily follows

$$\|(w(t), \dot{w}(t))\|^2_{\mathscr{H}^0} \leq C \left(\|(w_0, w_1)\|^2_{\mathscr{H}^0} + \|F\|^2_{L^2_T L^2(\Omega)} \right) e^t \quad \textit{for all} \quad t \in [0, T] \,.$$

Moreover, if $(w_0, w_1) \in \mathscr{H}^1$ *and* $F \in C^1_T L^2(\Omega)$*, then* $w \in C^2_T L^2(\Omega)$ *and* $w_{xx} \in C_T L^2(\Omega)$ *(e.g. see [24, Thm. 2.1, p. 229][1])*

□

4.2 Variational formulation

Proposition 4.2 (Variational form)**.** *Suppose* u *is a weak solution of* (1)–(3) *on* $[0, T]$*. Then for any test-function* $v \in C^1_t L^2(\Omega) \cap C_t H^1_0(\Omega)$ *with* $t \in [0, T]$*, it satisfies the variational identity*

$$(\dot{u}, v)_0 \Big|_0^t - \int_0^t (\dot{u}, \dot{v})_0 + \int_0^t (u, v)_1 + \int_0^t (f(u)\dot{u}, v)_0 = 0 \tag{24}$$

Proof. Let $c \in C^1([0, T])$, then for any $\phi \in H^1_0(\Omega)$ we have from (12)

$$(\dot{u}(t), c(t)\phi)_0 \Big|_0^t - \int_0^t (\dot{u}(t), c'(t)\phi)_0 dt + \int_0^t (u(t), c(t)\phi)_1 dt + \int_0^t (f(u(t))\dot{u}(t), c(t)\phi)_0 \, dt = 0 \,. \tag{25}$$

Let $\{E_n\}$ be the orthonormal basis for $L^2(\Omega)$ consisting of the eigenfunctions of A. Given $v \in C^1_t L^2(\Omega) \cap C_t H^1_0(\Omega)$ we can represent it as

$$v(s, x) = \sum_{k=1}^{\infty} c_k(s) e_k(x)$$

for $c_k \in C^1([0, T])$, $k \in \mathbb{N}$. This series that converges to v in $C_t H^1_0(\Omega)$ and its time-derivative $\sum_{k=1}^{\infty} c'_k(s) e_k(x)$ converges to v' in $C_t L^2(\Omega)$.

By applying identity (25) to finite sums $\sum^M c_k e_k$ and passing to the limit $M \to \infty$ we recover (24). □

[1] There's a minor misprint in [24, Thm. 2.1, p. 229]: the first assumption is meant to read $u_0 \in H^1_0(\Omega)$ (instead of "$H^2_0(\Omega)$"). In the second half of that theorem, which is the one we cite, it is strengthened to $u_0 \in H^1_0(\Omega) \cap H^2(\Omega)$ for strong solutions.

4.3 Relation between regularity and higher-order energy

The existence of finite energy solutions to (1)–(3) is known. Well-posedness for *regular solutions*, however, requires more work and relies on the connection between the smoothness in space and smoothness in time as summarized by the diagram below:

$$E_u^{(n)}(t) := \frac{1}{2}|\partial_t^n u(t)|_1^2 + \frac{1}{2}|\partial_t^n \dot{u}(t)|_0^2 .$$

$$\begin{array}{ccc} \text{local solutions in } C_T\mathscr{H}^n & \longrightarrow & E_u^{(n)}(t) \text{ is well defined} \\ \uparrow & & \downarrow \\ \text{bound on the } \mathscr{H}^n \text{ norm} & \longleftarrow & \text{Energy identity and a bound on } E_u^{(n)}(t) \end{array}$$

The purpose of this subsection is to furnish this connection which can be loosely outlined as follows: a weak solution u of (1)–(3) is regular of order n on $[0,T]$ if and only if the n-th order energy is bounded on $[0,T]$. We split this claim into two propositions.

In order not to keep track of how the structure of $f(u)\dot{u}$ changes after differentiation we introduce the following, somewhat abstract property:

Definition 4.1 (Regularity dependence)**.** *We say two functions (z,F) have **order** n **dependence** if for every $k = 1,2,\ldots n$ the regularity (**if** it holds)*

$$z \in C_T^j \mathscr{D}(A^{\frac{n+1-j}{2}}) \quad \textit{for} \quad j \le k$$

***would imply** that*

$$F \in C_T^j \mathscr{D}(A^{\frac{n-j}{2}}) \quad \textit{for} \quad j \le k$$

where j is a non-negative integer. It is helpful to note:

- *if (z,F) have order n dependence, they trivially have order ℓ dependence for any $\ell = 0,1,2,\ldots,n$.*
- *if (z,F) have order n dependence, then for $s \in \mathbb{N}$, $s \le n$ the functions $A^{s/2}z$ and $A^{s/2}F$ have order $(n-s)$ dependence.*

Again, it is helpful to consider an example.

Example 4.1. *Let $F = f(z)\dot{z}$. Then it is not hard to check that (z,F) have order n dependence. For instance take $n = 4$ and assume $z \in C_T^j\mathscr{D}(A^{\frac{5-j}{2}})$. As was shown via (17) and (18) in Example 3.1*

$$\partial_t^2 F \in \mathscr{D}(A)$$

continuously in time. In particular, $F \in C_T^2\mathscr{D}(A^{\frac{4-2}{2}})$.

Proposition 4.3. *Suppose that u is a weak solution on $[0,T]$ to $[\![F,u_0,u_1]\!]$ with $F \in C_TL^2(\Omega)$ and $\|(u,\dot{u})\|_{C_T\mathscr{H}^0} \le R_1$. Assume functions (u,F) have order n dependence for some $n \in \mathbb{N}\cup\{0\}$.*

Suppose, in addition,

$$(\partial_t^n u, \partial_t^n \dot{u}) \in C_T\mathscr{H}^0 \tag{26}$$

with $\|(\partial_t^n u, \partial_t^n \dot{u})\|_{C_T\mathscr{H}^0} \le R_2$, then

$$u \in S_T^n, \tag{27}$$

in other words, $u \in C^j_T \mathscr{D}(A^{\frac{n+1-j}{2}})$ *for* $j \le n+1$*. Moreover,*

$$\|(u,\dot{u})\|_{S^n_T} \le K(n, R_1, R_2), \tag{28}$$

for a constant $K(n,R_1,R_2)$ *dependent only on* n*,* R_1*,* R_2 *and being continuous monotone increasing with respect to the parameters* R_1*,* R_2*.*

Proof. Throughout the argument below, the norms in the considered spaces can be (inductively) estimated in terms of R_1 and R_2, thus ultimately verifying (28). We will focus in detail on proving the claimed regularity.

Cases $n = 0$**,** $n = 1$**.** By assumption we always have $(u,\dot{u}) \in C_T\mathscr{H}^0$ and $F \in C_T L^2(\Omega)$ which takes care of the case $n = 0$. If we in addition assume $(\dot{u},\ddot{u}) \in C_T\mathscr{H}^0$, then the equation $\ddot{u} - u_{xx} = F$ implies that $u_{xx} \in C_T L^2(\Omega)$. Since $u = 0$ on the boundary then $u \in C_T\mathscr{D}(A)$. Conclude:

$$(u,\dot{u}) \in S^1_T\,.$$

For $n \ge 2$ proceed by induction: suppose the result of this Proposition holds for $0, 1, \ldots, n-1$. Assume (26) holds. Then let us show (27).

Case $n \ge 2$**.** Because $(u,\dot{u}) \in C_T\mathscr{H}^0$, then condition (26) implies $(u,\dot{u}) \in C^n_T\mathscr{H}^0$. As a special case (using $n-1$ instead of maximal n), we have

$$(\partial_t^{n-1}u, \partial_t^{n-1}\dot{u}) \in C_T\mathscr{H}^0\,.$$

Moreover, functions (u,F) have a fortiori $(n-1)$-st order dependence, so the induction hypothesis gives

$$u \in S^{n-1}_T\,.$$

Next, by assumption (26), we also have

$$\partial_t^n \dot{u} = \partial_t^{n+1}u \in C_T L^2(\Omega)\,. \tag{29}$$

To show that $u \in S^n_T$, it remains to verify that for $j = 0, \ldots, n$ we have $\partial_t^j u \in C_T\mathscr{D}(A^{\frac{n+1-j}{2}})$. To this end introduce

$$z = A^{1/2}u \quad \text{and} \quad \tilde{F} = A^{1/2}F.$$

Since (u,F) have n-th order dependence, then z and $\tilde{F}$ have $(n-1)$-st order dependence (see Definition 4.1). As we already know, $u \in S^{n-1}_T$, and via the $(n-1)$-st order dependence of (u,F), we have $F \in C^j_T\mathscr{D}(A^{\frac{n-1-j}{2}})$ for $j \le n-1$. Because $n \ge 2$, then $A^{1/2}F \in C_T L^2(\Omega)$; likewise $n \ge 2$ also tells us that u is at least in $C_T\mathscr{D}(A)$, so $A^{1/2}u_{xx} = A^{3/2}u = \partial_{xx}A^{1/2}u \in C_T\mathscr{D}(A^{-1/2})$.

Consequently, applying $A^{1/2}$ to the equation for u, we conclude that $z = A^{1/2}u$ is a weak solution to $[\![\tilde{F}, z(0), \dot{z}(0)]\!]$ with $\tilde{F} \in C_T L^2(\Omega)$ and $(z(0),\dot{z}(0)) \in \mathscr{H}^0$. By (26) we also have

$$(\partial_t^{n-1}z, \partial_t^{n-1}\dot{z}) \in C_T\mathscr{H}^0\,.$$

The induction hypothesis now states that

$$z \in S^{n-1}_T \quad \text{for} \quad j = 0, 1, \ldots, n\,.$$

It implies that

$$u \in C^j_T\mathscr{D}(A^{\frac{n+1-j}{2}}) \quad \text{for} \quad j = 0, 1, \ldots, n,$$

as desired. From here, along with (29), it follows that $u \in S^n_T$ which completes the proof of the implication "case $n-1 \Rightarrow$ case n" for the induction argument. $\square$

The next result complements the previous proposition, demonstrating that the same regularity S_T^n of solutions can be inferred from a priori differentiability in space, rather than in time.

Proposition 4.4. *Suppose u is a weak solution on $[0,T]$ of $[\![F,u_0,u_1]\!]$ with $F \in C_T L^2(\Omega)$. Assume (u,F) have n-th order dependence (Definition 4.1) for some $n \in \mathbb{N} \cup \{0\}$. If u is regular of order n, i.e., $\|(u,\dot{u})\|_{C_T\mathscr{H}^n} \leq R < \infty$, then*

$$u \in S_T^n$$

and

$$\|(u,\dot{u})\|_{S_T^n} \leq K(n,R) \tag{30}$$

with K dependent only on n and R and continuous monotone increasing with respect to parameter R.

Proof. In the course of the proof, the bound (30) can be traced inductively to depend only on $\|(u,\dot{u})\|_{C_T\mathscr{H}^n}$. To keep the exposition concise the argument will focus on the regularity.

If $n=0$, the claimed regularity simply matches that of weak solutions. For $n=1$ we are given $u \in C_T\mathscr{D}(A)$ and $\dot{u} \in C_T\mathscr{D}(A^{1/2})$. Solving $\ddot{u} = u_{xx} - F \in C_T L^2(\Omega)$ verifies that $u \in C_T\mathscr{D}(A) \cap C_T^1 L^2(\Omega) = S_T^1$.

Proceed by induction. Fix $n \geq 2$, suppose the statement holds for all $\tilde{n} \leq n-1$ and assume $\|(u,\dot{u})\|_{C_T\mathscr{H}^n}$ is finite. A fortiori it follows that $(u,\dot{u}) \in C_T\mathscr{H}^{n-1}$. Then $u \in S_T^{n-1}$ by the induction hypothesis.

Define $z := A^{1/2}u$, then using $u \in S_T^{n-1}$ it follows as in the proof of Proposition 4.3 that z is a weak solution to $[\![A^{1/2}F, z(0), \dot{z}(0)]\!]$ with $A^{1/2}F \in C_T L^2(\Omega)$. Moreover from the assumption $u \in C_T\mathscr{H}^n$ we also have that $z \in C_T\mathscr{H}^{n-1}$, i.e., is regular of order $n-1$. Thus by the induction hypothesis

$$z \in S_T^{n-1}\,,$$

equivalently

$$u \in C_T^j\mathscr{D}(A^{\frac{n+1-j}{2}}) \quad \text{for} \quad j = 0,1,\ldots,n. \tag{31}$$

The only remaining step from here to proving $u \in S_T^n$ is to show that $\partial_t^{n+1}u \in C_T L^2(\Omega)$. To this end define

$$w := \partial_t^{n-1}u\,.$$

Then by (31)

$$(w,\dot{w}) \in C_T\mathscr{H}^1\,.$$

Since $u \in S_T^{n-1}$ and by the $(n-1)$-st order dependence (implied by n-th order dependence) of u and F, we deduce from (31) that $\partial_t^{n-1}F \in C_T L^2(\Omega)$. So

$$\ddot{w} = w_{xx} - \partial_t^{n-1}F \in C_T L^2(\Omega)$$

confirming that

$$\ddot{w} = \partial_t^{n+1}u \in C_T L^2(\Omega)$$

as desired. Thus $u \in S_T^n$ completing the induction argument. □

4.4 Local unique solutions

It is known that (1)–(3) possesses unique solutions [18]. Here we extend this result to regular solutions as well. First we formulate it for local in time solutions.

Theorem 4.1. *Suppose $\|(u_0, u_1)\|_{\mathscr{H}^n} = R < \infty$ for a non-negative integer n. Then there exists $T = T(R) > 0$ such that system (1)–(3) has a local unique solution that is regular of order n on interval $[0, T]$.*

Proof. **Step 1: the spaces.** Note that for $\psi \in \mathscr{D}(A^{n/2})$

$$\|\psi\|_{\mathscr{D}(A^{n/2})} \quad \lesssim \quad |\psi|_0 + |\partial_x^n \psi|_0 \, .$$

Moreover, for $(\psi_0, \psi_1) \in \mathscr{H}^n$ we have

$$\|\psi_0\|_{C^n(\overline{\Omega})} + \|\psi_1\|_{C^{n-1}(\overline{\Omega})} + \|\psi_1\|_{W^{n,2}(\Omega)} \quad \lesssim \quad \|(\psi_0, \psi_1)\|_{\mathscr{H}^n} \tag{32}$$

with the temporary notational convention $C^{-1}(\overline{\Omega}) := L^2(\Omega)$.

Step 2: contraction mapping. Let $t \mapsto \mathcal{S}(t)$ be the semigroup for the linear wave equation with data $y_0 \in \mathscr{H}^n$. Note that if z is regular of order n, then by Proposition 3.1 we would get

$$f(z)\dot{z} \in C_T \mathscr{D}(A^{n/2}) \, . \tag{33}$$

With this in mind introduce the operator Λ_T on $C_T\mathscr{H}^n$,

$$(\Lambda_T \phi)(t) := \mathcal{S}(t) y_0 + \int_0^t \mathcal{S}(t-s) \begin{bmatrix} 0 \\ f(\phi_0(s))\phi_1(s) \end{bmatrix} ds \quad \text{for any} \quad \phi = (\phi_0, \phi_1) \in C_T \mathscr{H}^n \, .$$

For $t \in [0, T]$ we have

$$\left\| \Lambda_T \phi(t) - \Lambda_T \tilde{\phi}(t) \right\|_{\mathscr{H}^n} \le \int_0^t \left\| \mathcal{S}(t-s) \begin{bmatrix} 0 \\ f(\phi_0(s))\phi_1(s) - f(\tilde{\phi}_0(s))\tilde{\phi}_1(s) \end{bmatrix} \right\|_{\mathscr{H}^n} ds \, . \tag{34}$$

From the local Lipschitz property (5) of f we obtain

$$\begin{aligned} f(\phi_0)\phi_1 - f(\tilde{\phi}_0)\tilde{\phi}_1 =& \Big[f(\phi_0)\phi_1 - f(\tilde{\phi}_0)\phi_1 \Big] + \Big[f(\tilde{\phi}_0)\phi_1 - f(\tilde{\phi}_0)\tilde{\phi}_1 \Big] \\ =& \phi_1 M(\phi_0, \tilde{\phi}_0)(\phi_0 - \tilde{\phi}_0) + f(\tilde{\phi}_0)(\phi_1 - \tilde{\phi}_1) \, . \end{aligned}$$

Now we will rely on the fact that a priori $\phi, \tilde{\phi} \in C_T \mathscr{H}^n$ gives:

$$\phi_0, \ \tilde{\phi}_0 \in C_T C^n(\overline{\Omega}), \quad \phi_1, \ \tilde{\phi}_1 \in C_T C^{n-1}(\overline{\Omega}) \quad \text{and} \quad \phi_1, \tilde{\phi}_1 \in C_T W^{2,n}(\Omega) \, .$$

Estimate,

$$\begin{aligned} & \left\| f(\phi_0)\phi_1 - f(\tilde{\phi}_0)\tilde{\phi}_1 \right\|_{\mathscr{D}(A^{n/2})} \\ \lesssim & \left| \phi_1 M(\phi_0, \tilde{\phi}_0)(\phi_0 - \tilde{\phi}_0) + f(\tilde{\phi}_0)(\phi_1 - \tilde{\phi}_1) \right|_0 + \left| \partial_x^n \left[\phi_1 M(\phi_0, \tilde{\phi}_0)(\phi_0 - \tilde{\phi}_0) + f(\tilde{\phi}_0)(\phi_1 - \tilde{\phi}_1) \right] \right|_0 \\ \le & \left| \phi_1 M(\phi_0, \tilde{\phi}_0)(\phi_0 - \tilde{\phi}_0) + f(\tilde{\phi}_0)(\phi_1 - \tilde{\phi}_1) \right|_0 \\ & + \left| \sum_{i+j+k=n} c_{ijk} \, \partial_x^i \phi_1 \cdot \partial_x^j M(\phi_0, \tilde{\phi}_0) \cdot \partial_x^k (\phi_0 - \tilde{\phi}_0) \right|_0 + \left| \sum_{i+j=n} d_{ij} \, \partial_x^i f(\tilde{\phi}_0) \cdot \partial_x^j (\phi_1 - \tilde{\phi}_1) \right|_0 \\ \le & \mathcal{P}_1 \left\{ \|\phi_0\|_{C^n(\overline{\Omega})}, \|\tilde{\phi}_0\|_{C^n(\overline{\Omega})} \right\} \cdot \|\phi_1\|_{W^{n,2}(\Omega)} \cdot \|\phi_0 - \tilde{\phi}_0\|_{C^n(\overline{\Omega})} + \mathcal{P}_2 \left\{ \|\tilde{\phi}_0\|_{C^n(\overline{\Omega})} \right\} \cdot \|\phi_1 - \tilde{\phi}_1\|_{W^{2,n}(\Omega)} \end{aligned} \tag{35}$$

where $\mathcal{P}_i\{\cdot\}$ are a polynomials in the indicated variables with positive coefficients. Now, from (32)

$$\|\phi_0 - \tilde{\phi}_0\|_{C^n(\overline{\Omega})} + \|\phi_1 - \tilde{\phi}_1\|_{W^{2,n}(\Omega)} \lesssim \|\phi - \tilde{\phi}\|_{\mathscr{H}^n} .$$

So continuing (35) we get

$$\begin{aligned}\left\|f(\phi_0)\phi_1 - f(\tilde{\phi}_0)\tilde{\phi}_1\right\|_{\mathscr{D}(A^{n/2})} \leq & \mathcal{P}_3\left\{\|\phi_0\|_{C^n(\overline{\Omega})}, \|\tilde{\phi}_0\|_{C^n(\overline{\Omega})}, \|\phi_1\|_{W^{n,2}(\Omega)}\right\}\|\phi - \tilde{\phi}\|_{\mathscr{H}^n} \\ \leq & \mathcal{P}_4\left\{\|\phi\|_{\mathscr{H}^n}, \|\tilde{\phi}\|_{\mathscr{H}^n}\right\}\|\phi - \tilde{\phi}\|_{\mathscr{H}^n} .\end{aligned}$$

Because the group $\mathcal{S}(t)$ is unitary on $\mathscr{D}(\mathbb{A}^n)$ then (34) yields

$$\left\|\Lambda_T\phi - \Lambda_T\tilde{\phi}\right\|_{C_T\mathscr{H}^n} \leq T\mathcal{P}_4\left\{\|\phi\|_{C_T\mathscr{H}^n}, \|\tilde{\phi}\|_{C_T\mathscr{H}^n}\right\}\|\phi - \tilde{\phi}\|_{C_T\mathscr{H}^n} . \tag{36}$$

If ϕ and $\tilde{\phi}$ come from a bounded set, then choosing T small enough implies that Λ_T is a contraction.

Step 3: invariance. Finally, we also need Λ_T to map the the ball $B_R(0)$ in $C_T\mathscr{H}^n$ into itself. According to (36) for that it suffices that R satisfy

$$T\mathcal{P}_4\{R, R\}2R \leq R .$$

For which we can impose

$$T < 1/(2\mathcal{P}_4\{R, R\}) . \tag{37}$$

Now the contraction mapping principle implies the claimed local unique solvability. □

4.5 Energy identities and global existence

The global existence result is based on a priori bounds on the energy.

Proposition 4.5 (Extension of local solutions)**.** *Let f be as in* (4) *with exponent $2m$, $m \in \mathbb{N}$. Assume u is a weak solution that is regular of order n, defined on some right-maximal interval $I_{max} = [0, T_{max})$. Suppose $(u, \dot{u}) \in C_T^n\mathscr{H}^0$ and that there exists a continuous function ψ on $\mathbb{R}_+$ such that*

$$E_u^{(n)}(t) \leq \psi(t) \quad on \quad I_{max} .$$

Then $T_{\max} = \infty$.

Proof. By Proposition 3.1, u and the term $F = f(u)\dot{u}$ have n-th order dependence. Recall that energy $E_u^{(n)}$ controls the $C_T\mathscr{H}^n$ norm of the solution (in fact, it controls the $L_T^\infty\mathscr{H}^n$ norm, but the continuity with values in $\mathscr{H}^n$ is implied by the definition of regular solution). Then there is a constant K dependent on $\sup\{|\psi(t)| : 0 \leq t \leq T_{\max}\}$ such that

$$\|(u, \dot{u})\|_{C_T\mathscr{H}^n} \leq K$$

for any $T < T_{\max}$. Hence by Theorem 4.1 any initial data of the form $v_0 = u(t)$ and $v_1 = \dot{u}(t)$ for $t \in [0, T_{\max})$ can be extended to a solution that exists for another $\Delta T = \Delta T(K)$ time units, independently of $t \in [0, T_{\max})$. Hence $T_{\max}$ cannot be finite. □

Lemma 4.2 (Energy identity for regular solutions)**.** *Let f be as in (4) with exponent $2m$, $m \in \mathbb{N}$. Suppose u is a weak solution to (1)–(3) on $[0,T]$. If u is regular of order n, then for $k \le n$ we have $(u,\dot u) \in C^k_T \mathscr{H}^0$, $f(u)\dot u \in C^k_T L^2(\Omega)$ and*

$$E_u^{(k)}(t) + \int_0^t (\partial_t^k[f(u)\dot u], \partial_t^k \dot u)_0 = E_u^{(k)}(0)\,. \tag{38}$$

Proof. Proposition 3.1 confirms that F and $F = f(u)\dot u$ have n-th order regularity dependence. Invoke Proposition 4.3 to conclude that $u \in S^n_T$. It certainly implies that $(u,\dot u) \in C^k \mathscr{H}^0$ for $k \le n$. And by the n-th order regularity dependence of (u,F) we also have $F \in C^k_T L^2(\Omega)$.

Define

$$w = \partial_t^k u$$

then w is a weak solution to $[\![\partial_t^k F, w(0), \dot w(0)]\!]$ with $\partial_t^k F \in C_T L^2(\Omega)$ and $(w(0), \dot w(0)) \in \mathscr{H}^0$. Now call upon the energy identity (23) for w to obtain (38) for u. □

4.5.1 Finishing the proof: global existence and regularity

Let u be a weak solution on interval $[0,T]$. Because $F \cdot \dot u := f(u)\dot u^2$ is non-negative by assumption (4) on f, then in the case $n = 0$ by Lemma 4.2 we have

$$\tfrac{1}{2}\|(u,\dot u)\|^2_{C_T\mathscr{H}^0} \le \sup_{t\in[0,T]} E_u(t) \le E_u(0)\,. \tag{39}$$

Since the bound is independent of $T \in [0, T_{\max})$, then by Proposition 4.5 u extends globally to $t \ge 0$.

Arguing by induction, suppose that $(u,\dot u)$ is a global solution that is regular of order n, and at the same time is *local* regular of order $n+1$ on a maximal interval $[0, T_{\max})$. We want to show that it is also global of order $n+1$.

Example 4.2. *It helps to preview the result on the example of $f(s) = s^2$ and by extending from weak to regular of order 1 solutions. Suppose u is a global weak solution that is, in addition, regular of order 1 on maximal interval $[0,T_{\max})$. For such a solution by Proposition 4.2 we have the "1-st order" energy identity*

$$E_u^{(1)}(T) + \int_0^T \partial_t[f(u)\dot u]\ddot u = E_u^{(1)}(0)\,.$$

$$E_u^{(1)}(T) + \int_0^T (2u\dot u^2 + u^2\ddot u)\ddot u = E_u^{(1)}(0)\,.$$

We can bound $C_T L^\infty(\Omega)$ norms of u and $\dot u$ in terms of the finite energy $E_u(t)$. The latter is bounded by constant $E_u(0)$ (but, in fact, any continuous on $\mathbb{R}_+$ upper bound on $E_u(t)$ would do, which is how the general argument works). In turn, the term $|\ddot u|_0^2$ is a part of the first-order energy. So we have

$$E^{(1)}(T) \le E_u^{(1)}(0) + C(E_u^{(0)}(0)) \int_0^T E_u^{(1)}(t)dt$$

From here, Gronwall's estimate gives us an asymptotically growing continuous upper bound on $E_u^{(1)}(t)$. Then Proposition 4.5 ensures that u is regular of order 1 globally, that is $T_{\max} = \infty$.

Now onto the actual proof of global existence. By the n-th order global regularity there is a continuous monotone increasing function $\psi_n : \mathbb{R}_+ \to \mathbb{R}_+$

$$\|(u, \dot{u})\|_{S_t^n} \leq \psi_n(t) \quad \text{for all} \quad t \geq 0\,.$$

By $(n+1)$'st order regularity and via Lemma 4.2 for any $T \leq [0, T_{\max})$ we also have

$$E_u^{(n+1)}(T) + \int_0^T (\partial_t^{n+1} F, \partial_t^{n+1}\dot{u})_0 = E_u^{(n+1)}(0)\,.$$

Use estimate (15) of Proposition 3.1 to claim that there is a polynomial $\mathcal{P}(s)$ such that for each t

$$|\partial_t^{n+1} F(t)|_0 \leq \mathcal{P}(\psi_n(t))(1 + |\partial_t^{n+1}\dot{u}(t)|_0)\,.$$

Then $|\partial_t^{n+1} F(t)|_0 |\partial_t^{n+1}\dot{u}|_0 \leq \mathcal{P}(\psi_n(t))(1 + |\partial_t^{n+1}\dot{u}|_0 + |\partial_t^{n+1}\dot{u}|_0^2) \leq 2\mathcal{P}(\psi_n(t))(1 + |\partial_t^{n+1}\dot{u}|_0^2)$. In particular,

$$E_u^{(n+1)}(T) \leq E_u^{(n+1)}(0) + 4\int_0^T \mathcal{P}(\psi_n(t))(1 + E_u^{(n+1)}(t))dt\,.$$

Gronwall's inequality now provides

$$E_u^{(n+1)}(T) \leq \psi_{n+1}(T) := \left(E_u^{(n+1)}(0) + \int_0^T 4\mathcal{P}(\psi_n(t))dt\right)\exp\left(\int_0^T 4\mathcal{P}(\psi_n(t))dt\right) \quad \text{for all} \quad T < T_{\max}\,.$$

By the assumption on ψ_n, the newly defined mapping ψ_{n+1} is continuous on $\mathbb{R}_+$. Then Proposition 4.5 ensures that the solution u is global, regular of order $n+1$, which completes the proof by induction.

This step also completes the proof of Theorem 3.1. The additional regularity $u \in S_T^n$ follows if we first use Proposition 3.1, which establishes n-th order dependence of u and $f(u)\dot{u}$, and then invoke Proposition 4.3 which translates n-th order regularity into S_T^n.

5 Lack of uniform stability

This section is devoted to the proof of Theorem 3.2. The heart of the argument is that the energy decay slows down with the frequency of solutions, independently of their total finite energy.

Let $k \in \mathbb{N}$ and consider initial condition of the form

$$u_0^{(k)}(0,x) = \frac{2}{k\pi}\sin(k\pi x), \qquad \partial_t u_1^{(k)}(0,x) = 0\,. \tag{40}$$

Note that $u_0^{(k)} \in \mathscr{D}(A^r)$ for any $r \in \mathbb{R}$. By direct calculation we have

$$E_{u^{(k)}}(0) = \tfrac{1}{2}\|\partial_x u_0^{(k)}\|^2 + \tfrac{1}{2}\|u_t^{(k)}(0)\|^2 = \tfrac{1}{2}\|\partial_x u_0^{(k)}\|^2 = 1 \quad \text{and} \quad \|u_0^{(k)}\| = \frac{\sqrt{2}}{k\pi}\,. \tag{41}$$

Hence for such initial data, any constant that may be estimated in terms of $\|(u_0^{(k)}, u_1^{(k)})\|_{\mathscr{H}^0}$ is bounded uniformly in k.

5.1 "Primitive" problem and decay of lower-order norms

We continue working with the initial data $(u_0^{(k)}, 0)$ as in (40). Pick the anti-derivative through the origin of f (as in (4))

$$\tilde{f}(s) := \frac{s^{2m+1}}{2m+1} \,.$$

Due to $|u_0^{(k)}|_1^{2m}$ (hence $\|u_0^{(k)}\|_{C(\overline{\Omega})}^{2m}$) being bounded independently of k, we have for some constant C **independent of** k **in** (40) the estimate

$$|\tilde{f}(u_0^{(k)})|_0 \leq C |u_0^{(k)}|_0 \,.$$

Let function $\Phi^{(k)}$ solve the linear elliptic boundary value problem

$$\begin{cases} \partial_{xx}\Phi^{(k)} &= \tilde{f}(u_0^{(k)}) \\ \Phi^{(k)}(0) &= \Phi^{(k)}(1) = 0 \,. \end{cases} \tag{42}$$

It satisfies the standard elliptic estimate:

$$|\Phi^{(k)}|_1 \leq C' |\tilde{f}(u_0^{(k)})|_0 \leq C |u_0^{(k)}|_0 \tag{43}$$

for C' and C **independent of** k. We do have $\tilde{f}(u_0^{(k)}) \in L^2(\Omega)$ because $u_0^{(k)} \in H_0^1(\Omega)$. In fact, $u_0^{(k)}$ is an eigenfunction, and it is easy to show (as in Proposition 3.1) that

$$\tilde{f}(u_0^{(k)}) \in \mathscr{D}(A) \cap C^\infty(\Omega) \,.$$

We then have

$$\Phi^{(k)} \in \mathscr{D}(A^2) \subset W^{4,2}(\Omega) \,.$$

With this function $\Phi^{(k)}$ at hand, consider the nonlinear "primitive" problem

$$\ddot{\phi}^{(k)} - \phi_{xx}^{(k)} + \tilde{f}(\dot{\phi}^{(k)}) = 0 \quad \text{for} \quad x \in \Omega, \ t > 0 \tag{44}$$

with homogeneous Dirichlet boundary data

$$\phi^{(k)}(t, 0) \equiv \phi^{(k)}(t, 1) \equiv 0 \quad \text{for} \quad t > 0 \tag{45}$$

and initial condition

$$\phi^{(k)}(0) = \Phi^{(k)} \quad \text{and} \quad \dot{\phi}^{(k)}(0) = u_0 \,. \tag{46}$$

Note the following properties of the system (44)–(46):

i.) Because $s \mapsto \tilde{f}(s)$ is continuous, monotone increasing vanishing at 0, then (44) is a semilinear wave problem with monotone damping. The data $(\Phi^{(k)}, u_0^{(k)})$ come from the domain of the associated nonlinear generator; in this 1D setting it coincides with the domain $\mathscr{H}^1$ of the corresponding linear group. This well-posedness result is based on monotone operator theory [25, Thm. 3.7, p. 306].

In particular, we have for any $T > 0$

$$\phi^{(k)} \in L_T^\infty \mathscr{D}(A) \cap C_T^1 \mathscr{D}(A^{1/2}) \quad \text{and} \quad \ddot{\phi}^{(k)} \in L_T^\infty L^2(\Omega) \,.$$

Moreover, given the 1D embeddings it follows that

$$\tilde{f}(\dot{\phi}^{(k)}) \in C_T^1 L^2(\Omega) \cap C_T \mathscr{D}(A^{1/2}) .$$

So $\phi^{(k)}$ satisfies (44) and has $(\phi(0), \dot{\phi}(0)) \in \mathscr{D}(A^2) \times \mathscr{D}(A^{3/2})$. By Proposition 4.1 we have

$$\phi_{xx}^{(k)},\ \ddot{\phi}^{(k)} \in C_T L^2(\Omega) .$$

ii.) The velocity $v^{(k)}(t,x) := \dot{\phi}^{(k)}(t,x)$ satisfies $(v^{(k)}, \dot{v}^{(k)}) \in L_T^2 \mathscr{H}^0$ and is a weak solution to

$$\ddot{v^{(k)}} - v_{xx}^{(k)} + f(v^{(k)}) v_t^{(k)} = 0 . \tag{47}$$

In addition,

$$v^{(k)}(0) = \dot{\phi}^{(k)}(0) = u_0^{(k)} \in L^2(\Omega) \qquad \text{(actually, in any } \mathscr{D}(A^r))$$

and because $\phi_{xx}^{(k)}$, $\ddot{\phi}^{(k)}$ are continuous with values in $L^2(\Omega)$, then

$$\dot{v}^{(k)}(0) \;=\; \ddot{\phi}^{(k)}(0) \overset{(44)}{=} \phi_{xx}^{(k)}(0) - \tilde{f}(\dot{\phi}^{(k)}) = \partial_x^2 \Phi^{(k)} - \tilde{f}(u_0^{(k)}) = 0$$

according to the way Φ_0 was constructed in (42). *Thus the primitive problem is the velocity potential for our original problem. Namely the time derivative $\dot{\phi}^{(k)}$ is the solution $u^{(k)}$ to our original degenerately damped system with initial data $(u_0^{(k)}, 0)$.*

iii.) Solutions to (44)–(46) decay uniformly to zero at the rate that can be estimated explicitly in terms of the exponent $2m$ of f. This is a fundamental example of a dissipative problem with full interior damping. The decay rate can be assessed using, for example, the ODE characterization of [22] (see [3, Coro. 1, p. 1770] for more details; in particular function $h(s)$ there must satisfy $h(s^{2m+2}) \geq s^2$). We get, as $t \to \infty$ that

$$E_{\phi^{(k)}}(t) \sim \left(\frac{1}{t}\right)^{1/m} .$$

It follows that the (lower-order) norm $|u^{(k)}(t)|_0^2 = |\dot{\phi}^{(k)}(t)|_0^2$ decays to zero as $t \to \infty$ with the decay rate uniform with respect to the finite energy $E_{u^{(k)}}(0)$. By interpolation between $L^2(\Omega)$ and $H^1(\Omega)$ we conclude uniform a decay (but at slower rates, where $1/m$ is modified by the interpolation exponent) for every norm $\|u^{(k)}(t)\|_{W^{s,2}(\Omega)}$ with $s \in (0,1)$.

Remark 5.1. *Note that uniform stability of the original system would have required us to include the case $s = 1$, which as we are about to show cannot happen.*

Summarizing: $v^{(k)} = \dot{\phi}^{(k)}$ is precisely the solution $u^{(k)}$ to $[\![f(u^{(k)})\dot{u}^{(k)}, u_0^{(k)}, 0]\!]$. Any fractional $W^{s,2}(\Omega)$ Sobolev norm of $u^{(k)}$ for $s \in (0,1)$ decays uniformly with respect to $E_{u^{(k)}}(0)$. In addition, we have the estimate

$$|u^{(k)}(t)|_0^2 = |\dot{\phi}^{(k)}(t)|_0^2 \leq 2E_{\phi^{(k)}}(t) \leq 2E_{\phi^{(k)}}(0) = |\partial_x \Phi^{(k)}|_0^2 + |u_0^{(k)}|_0^2 \overset{(43)}{\leq} C_2 |u_0^{(k)}|_0^2 \tag{48}$$

for C_2 independent of $u_0^{(k)}$, hence **independent of** k in (40). In particular, the lower order norm $|u^{(k)}(t)|_0$ of the solution to our original problem $[\![f(u)\dot{u}^{(k)}, u_0^{(k)}, u_1^{(k)}]\!]$ is controlled by its initial lower-order norm $|u_0^{(k)}|_0$ **independently of** k.

Now we are going to use the smallness of the $L^2(\Omega)$ norm of $u^{(k)}(t)$ to show that its $\mathscr{D}(A^{1/2})$ norm cannot decay too fast.

5.2 Comparison with a conservative problem

Let $w^{(k)}$ be the unique solution to *linear homogeneous* problem $[\![0, u_0^{(k)}, 0]\!]$. From the energy identity (23) and by the choice of $u_0^{(k)}$ in (40) we get

$$E_{w^{(k)}}(t) \equiv 1 = E_{u^{(k)}}(0)\,.$$

Define

$$z^{(k)} = u^{(k)} - w^{(k)}$$

then $z^{(k)}$ solves $[\![f(u^{(k)})\dot{u}^{(k)}, 0, 0]\!]$. The energy identity (23) for z gives

$$E_{z^{(k)}}(t) = \underbrace{E_{z^{(k)}}(0)}_{=0} - \int_0^t (f(u^{(k)})\dot{u}^{(k)}, \dot{z}^{(k)})_0 \quad \leq \quad \int_0^t \|f(u^{(k)})\|_{L^\infty(\Omega)} |\dot{u}^{(k)}|_0 |\dot{z}^{(k)}|_0$$

Note that

- $|\dot{u}^{(k)}|_0 \cdot |\dot{z}^{(k)}|_0 \leq C(\sqrt{E_{u^{(k)}}(0)E_{z^{(k)}}(0)}) \leq 2C$ (since $E_{u^{(k)}}(0) = 1$) **independently of** k in (40).
- Given f as in (4) we have $\|f(u^{(k)})\|_{L^\infty(\Omega)} \lesssim \|u^{(k)}\|^{2m}_{L^\infty(\Omega)}$. Via 1-dimensional embeddings and interpolation for $\varepsilon s < 1/2$

  $$\|u^{(k)}\|_{L^\infty(\Omega)} \lesssim \|u^{(k)}\|_{W^{1-\varepsilon,2}(\Omega)} \lesssim |u^{(k)}|_0^\varepsilon \cdot |u^{(k)}|_1^{1-\varepsilon}$$

  so

  $$\|u^{(k)}\|^{2m}_{L^\infty(\Omega)} \leq \|u^{(k)}\|^{2m-1}_{L^\infty(\Omega)} \cdot |u^{(k)}|_0^\varepsilon \cdot |u^{(k)}|_1^{1-\varepsilon} \;\leq\; C(E_{u^{(k)}}(0), \varepsilon)\, |u^{(k)}|_0^\varepsilon$$

  Because $E_u(0) = 1$ independently of k in (40), then by (48) we get for any $t \geq 0$

  $$\|f(u^{(k)}(t))\|_{L^\infty(\Omega)} \lesssim \|u^{(k)}(t)\|^{2m}_{L^\infty(\Omega)} \leq C_3 |u_0^{(k)}|_0^\varepsilon.$$

  for C_3 **independent of** k in (40). At this point we finally expand the definition of u_0 to get (see (41))

  $$\|f(u^{(k)}(t))\|_{L^\infty(\Omega)} \leq C_{3,\varepsilon} \frac{1}{k^\varepsilon} \quad \text{for any} \quad \varepsilon < 1/2 \quad \text{and} \quad t \geq 0\,.$$

Plugging these observations into the estimate for $E_{z^{(k)}}(t)$ we obtain:

Lemma 5.1. *For $u_0^{(k)}(x) := 2(k\pi)^{-1}\sin(k\pi x)$ let $u^{(k)}$ denote the weak solution on $[0,T]$ to the problem $[\![f(u^{(k)})\dot{u}^{(k)}, u_0^{(k)}, 0]\!]$ and $w^{(k)}$ be the solution to linear homogeneous problem $[\![0, u_0^{(k)}, 0]\!]$. Then the difference $z^{(k)} - w^{(k)}$ solves $[\![f(u^{(k)})\dot{u}^{(k)}, 0, 0]\!]$ and for any $\varepsilon < 1/2$ it satisfies*

$$E_{z^{(k)}}(t) \leq C_\varepsilon T \frac{1}{k^\varepsilon} \quad \textit{for all} \quad t \in [0,T]\,,$$

*with C_ε **independent of** k.* □

5.3 Ruling out uniform stability

At this point for brevity let us suppress the superscript "(k)" stemming from the choice of the parameter in the definition of initial data. Let u, w, z be as in Lemma 5.1. Suppose for a moment that $E_z(t) \le \delta$ for some t. Then via $E_w(t) \equiv 1 = E_u(0)$ we have

$$\begin{aligned} E_u(t) = E_{w+z}(t) &= \tfrac{1}{2}|w_t(t) + z_t(t)|_0^2 + \tfrac{1}{2}|w_x(t) + z_x(t)|_0^2 \\ &= E_w(t) + E_z(t) + (w_t(t), z_t(t))_0 + (w_x(t), z_x(t))_0 \\ &\ge E_w(t) + E_z(t) - 4\sqrt{E_w(t)E_z(t)} \ge 1 - 4\sqrt{\delta}\,. \end{aligned} \tag{49}$$

To apply this estimate, pick any $\overline{T} > 0$ and find $\delta > 0$ such that $1 - 4\sqrt{\delta} > 1/2$ (e.g., if $\delta < 1/64$). Fix $\varepsilon < 1/2$, then there is $k = k(\delta)$ large enough so that for initial condition $(u_0 = 2(\pi k)^{-1}\sin(k\pi x), 0)$ yields a solution whose energy satisfies

$$E_z(t) \le C_\varepsilon \frac{\overline{T}}{k^\varepsilon} < \delta \quad \text{for} \quad t \in [0, \overline{T}]\,,$$

for C_ε as in Lemma 5.1. Consequently, by (49),

$$E_u(t) \ge \frac{1}{2} \quad \text{for all} \quad t \in [0, \overline{T}]\,.$$

However, the initial condition $(u_0, 0)$ had energy 1 (again, independently of k). Hence the family of initial conditions

$$\left\{ \left(u_0^{(k)} := \frac{2}{k\pi}\sin(k\pi x),\ u_1^{(k)} := 0 \right) : k \in \mathbb{N} \right\} \tag{50}$$

with the associated solutions $u^{(k)}$, resides in a bounded ball (of radius $\sqrt{2E_{u^{(k)}}(0)} = \sqrt{2}$) in $\mathscr{H}^0$, yet the corresponding solutions do not decay to zero uniformly in the topology of $\mathscr{H}^0$.

Thus the associated dynamical system on $\mathscr{H}^0$ is not uniformly stable. This argument demonstrates Theorem 3.2 for $\bar{r} = 1$ and $r = \frac{1}{2}$. The general case follows merely by attaching a factor of $\sqrt{r}$ to $u_0^{(k)}$ and choosing a potentially smaller δ in the last step of the argument.

5.4 Monotonic strong decay of the energy

For a weak solution u of $[\![f(u)\dot{u}, u_0, u_1]\!]$ with $(u_0, u_1) \in \mathscr{H}^0$ the functional

$$t \mapsto E_u(t)$$

is non-increasing. We cannot presently claim whether solutions decay to 0 strongly, however, it is possible to show that the energy $E_u(t)$ is *strictly* monotonically decreasing for non-trivial solutions.

Consider a weak solution u on $[0, T]$ and suppose $E_u(t) \equiv \text{const}$ on $(a, b) \subset [0, T]$, then from the energy identity (23) follows that

$$\int_a^b (u(t)^{2m}, \dot{u}^2(t))_0 dt = 0$$

Thus $u^m\dot{u} \equiv 0$ a.e. in $(a, b) \times \Omega$. Since it is equivalent to $\frac{1}{m+1}\partial_t(u^{m+1}) = 0$, then we conclude that $u^{m+1}(t_1) = u^{m+1}(t_2)$ in $L^2(\Omega)$ for a.e. t_1, t_2:

$$u^{m+1} = \text{const} \quad \text{in} \quad L^2(\Omega) \quad \text{a.e.} \quad t \in (a, b).$$

Moreover, since $u \in C_T H_0^1(\Omega) \hookrightarrow C([0,T] \times \overline{\Omega})$, then we have $u(t,x) = \pm u_0(x)$ for every $(t,x) \in (a,b) \times \Omega$. By the continuity of the solution this is only possible if $u \equiv u_0$ for $(t,x) \in (a,b) \times \Omega$. Then $\dot{u} \equiv 0$ and we arrive at an equilibrium solution which has to be trivial. This observation completes the proof of Theorem 3.2.

6 Numerical results

The strategy for the proof of instability was largely prompted by numerical observations described below.

6.1 Outline of the numerical approach

The numerical implementation presented here treats the case $m = 1$ of (4):

$$\ddot{u} - u_{xx} + u^2 \dot{u} = 0$$

with

$$u(t,0) \equiv u(t,1) \equiv 0$$

and given initial data

$$u(0,x) = u_0(x) \quad \text{and} \quad \dot{u}(0,x) = u_1(x)\,.$$

Solution was discretized in space via a Ritz-Galerkin method. The dynamic problem could be analyzed explicitly using a discretization in time and a Runge-Kutta scheme, though, rigorous justification of convergence becomes more delicate. Another approach is to approximate the successive approximations of Theorem 4.1 which, when exact, are guaranteed to converge, at least over small time intervals. The iterates correspond to *linear* inhomogeneous PDE problems that are resolved using a hybrid scheme:

(i) for relatively short times find solutions using an approximation of semi-discrete Ritz-Galerkin method by discretizing time-integrals in the variation of parameter formula. If the error in numerical integration is small, then this approach enjoys an explicit convergence estimate (for smooth solutions and over finite time intervals) essentially proportional to the space discretization parameter h.

(ii) for larger times, collect the last k-points of the semi-discrete approximation and resolve the rest of the iteration using a multi-step method (Adams-Bashforth).

Thus, we begin with some initial guess $(u_{[0]}, \dot{u}_{[0]})$ and proceed to solve inhomogeneous linear problems

$$\ddot{u}_{[k]} - \partial_{xx} u_{[k]} = F_{[k-1]} \tag{51}$$

on the space $\mathscr{H}^0 = H_0^1(\Omega) \times L^2(\Omega)$ with the forcing term from the preceding iteration

$$F_{[k-1]} = -u_{[k-1]}^2 \dot{u}_{[k-1]}\,.$$

The constants in the estimates (36) and (37) could potentially be determined explicitly in this case (by following the proof with specific α and m in the definition (4) of f) which would yield an

explicit bound on the Lipschitz constant "γ" of the contraction mapping in terms of T. In turn, given this constant $\gamma > 0$ if

$$\|u_{[k]} - u_{[k-1]}\|_{C_T \mathscr{H}^0} < \varepsilon$$

then the absolute error between k-th iteration and the true solution is no more than $\varepsilon\gamma/(1-\gamma)$.

If we use a Ritz-Galerkin scheme with element size h to find an approximate solution $u_{[k,h]}$ to linear inhomogeneous problem (51), then for h small, e.g. see [26, Thm. 13.1, p. 202], and for simplicity taking the initial conditions to be the more accurate Ritz projections of the initial data, we get

$$\|u_{[k]} - u_{[k,h]}\|_{C_T \mathscr{H}^0} \leq Ch \int_0^T |\ddot{u}_{[k]}(s)|_2 ds\,.$$

This estimate of course requires sufficiently regular solutions. As Theorem 3.1 and Example 3.2 show, in order to have the $L^2 W^{2,2}(\Omega)$ regularity on $\ddot{u}_{[k]}$ it suffices to have initial data in the space

$$(u(0), \dot{u}(0)) \in \mathscr{H}^2 = \mathscr{D}(A^2) \times \mathscr{D}(A^{3/2})\,.$$

For example, the demonstrated numerical results below use displacement and velocity proportional to the eigenfunctions of A, which are smooth.

6.2 Specifics of the implementation

Consider an equipartitioned mesh of subintervals of length h and the standard piecewise linear nodal basis $\{\phi_i^h\}$, with $i = 1, \dots N := (h^{-1} - 1)$. Let $\mathbb{A}_h$ denote the restriction of the linear evolution generator $\mathbb{A}$ to the subspace $\mathscr{H}_h^0$ of $\mathscr{H}$ spanned by $\{\psi_{ij}^{(h)} = (\phi_i^{(h)}, \phi_j^{(h)})\}$. By $\mathcal{R}_h^1$ denote the elliptic Ritz projection on the subspace of $H_0^1(\Omega)$ and let $\mathcal{R}_h^0$ stand for the corresponding $L^2(\Omega)$ projection.

Given a candidate approximation

$$y_{[k-1,h]} = \begin{bmatrix} u_{[k-1,h]} \\ \dot{u}_{[k-1,h]} \end{bmatrix}$$

we compute the forcing

$$f_{[k-1,h]}(t,x) = u_{[k-1,h]}^2(t,x)\dot{u}_{[k-1,h]}(t,x)$$

The coefficient vector $\mathbf{f}_{[k,h]}$ of the projection of $\mathcal{R}_h^0 f_{[k,h]}$ is obtained in terms of the coefficients $C_{pqrs} = \displaystyle\int_\Omega \phi_p^h(x)\phi_q^h(x)\phi_r^h(x)\phi_s^h(x)dx$ (which for this choice of basis functions ϕ_i^h form a very sparse tensor with only 3 distinct values). The initial guess used to calculate $f_{[0,h]}$ is the constant solution :

$$f_{[0,h]}(t,x) = (\mathcal{R}_h^1 u_0(x))^2 (\mathcal{R}_h^0 u_1(x)) \quad \text{for all} \quad t \geq 0.$$

Let $\mathbf{y}_0$ denote the projection of the initial data $(\mathcal{R}_h^1 u_0, \mathcal{R}_h^0 u_1)$. We obtain a semi-discrete approximation of the original system (1)–(3) for unknown coefficient vector $\mathbf{y}_{[k,h]}$:

$$\mathbf{y}'_{[k,h]} = -\mathbb{A}_h \mathbf{y}_{[k,h]} + \mathbf{f}_{[k-1,h]}, \qquad \mathbf{y}(0) = \mathbf{y}_0\,.$$

For relatively short times we can invoke

$$\mathbf{y}_{k,h}(t) = e^{-t\mathbb{A}_h}\mathbf{y}_0 + I_t \quad \text{with} \quad I_t := \int_0^t e^{-\mathbb{A}_h(t-s)}\mathbf{f}_{[k-1,h]}(s)ds$$

which is in turn discretized over time scale $\mathbf{T} = (t_1, t_2, \dots, t_N)$ with $t_{i+1} - t_i = \delta$. According to

$$I_t = e^{-\mathbb{A}_h(t-\bar{t})} I_{\bar{t}} + \int_{\bar{t}}^{t} e^{-\mathbb{A}_h(t-s)} \mathbf{f}_{[k-1,h]}(s) ds \,.$$

at each step only the integral over $[\bar{t}, t]$ needs to be computed. For this purpose only several values of the matrix exponentials $e^{-\mathbb{A}_h(t-s)}$ are needed in order to apply the Newton-Cotes rule on sub-interval $[\bar{t}, t]$, specifically:

$$e^{j \cdot \delta \mathbb{A}_h} \quad \text{for} \quad j = 1, 2, \dots, m-1$$

where m is number of points used for Newton-Cotes formula (e.g., Boole's or Simpson's 3/8th). These $m-1$ matrices need to be computed just once and only depend on the time-step, but not the total number of these steps.

In turn, the vectors $e^{-t_k \mathbb{A}_h} \mathbf{y}_0$ have to be found for each t_k. But since $\mathbf{y}_0$ is fixed, these can be more accurately determined using scaling and truncated Taylor series approximation [27].

For simulations over larger time-scales we can use the last few values:

$$(t_{N-p}, \mathbf{y}_{k,h}(t_{N-p})), \cdots, (t_N, \mathbf{y}_{k,h}(t_N))$$

to initialize a linear p-step method, e.g., 5-step Adams-Bashforth to efficiently obtain solution on the interval $[t_N, t_{\text{final}}]$.

6.3 Pointwise Runge-Kutta solutions for particular initial data

As before, let E_k be the eigenfunction $\sqrt{2}\sin(\pi k x)$ for A with eigenvalue $\lambda_k = \pi^2 k^2$. Then for initial data

$$u_0(x) = c_0 E_k(x) \quad \text{and} \quad u_1 = c_1 E_k(x) \tag{52}$$

for constants c_0, c_1, the solution of (1)–(3) can be reduced to a dissipative ODE using the ansatz

$$u(t, x) = \phi(t) E_k(x) \,. \tag{53}$$

Plugging it into (1) equation yields

$$\phi'' E_k + \lambda_k \phi E_k + E_k^2 \phi^2 \phi' = 0$$

This identity would be implied if for each $x \in \Omega$ function ϕ solves the 2nd-order nonlinear ODE

$$\phi'' + \lambda_k \phi + E_k(x) \phi^2 \phi' = 0 \,.$$

$$\phi(0) = c_1 \quad \text{and} \quad \phi'(0) = c_2.$$

It corresponds to a first-order nonlinear system:

$$\Psi' = \mathbb{F}_k(\Psi, x), \qquad \mathbb{F}_k\left(\begin{bmatrix}\alpha \\ \beta\end{bmatrix}, x\right) := \begin{bmatrix}\beta \\ -\lambda_k \alpha - E_k(x)\alpha^2\beta\end{bmatrix}, \quad \Psi(0) = \begin{bmatrix}c_0 \\ c_1\end{bmatrix} \,. \tag{54}$$

Function $\mathbb{F}_k$ is smooth with respect to the components of Ψ and to variable x, which now acts as a parameter. This ODE system has global differentiable solutions, moreover since $E_k(x)$ is smooth, in fact, analytic in x then local solutions are differentiable with respect to x [28, Thm. 3.1, p. 95].

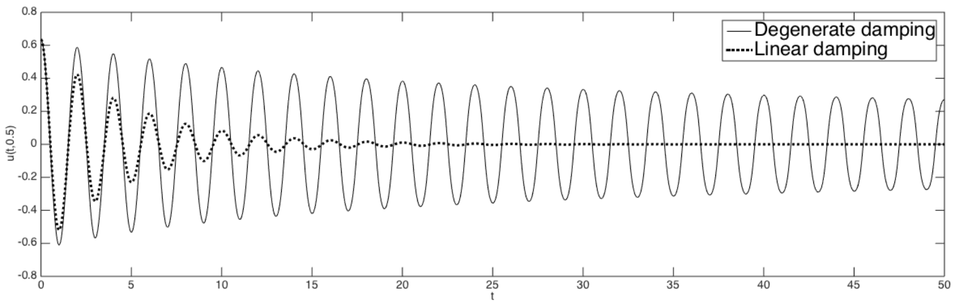

Figure 1: The displacement value $u(t, x = 0.5)$ of a numerical solution to problem (1)–(3) with $f(s) = s^2$. Initial displacement $u_0(x) = 2\pi^{-1}\sin(\pi x)$, initial velocity $u_1(x) = 0$. Time interval: $[0, T = 10]$. Obtained by approximating the Ritz-Galerkin semi-discrete solution using numerical integration in the variation of parameter formula. Element size: $h = 10^{-2}$; time-step: $\delta = 2 \cdot 10^{-3}$. Displayed next to the exact analytic solution of the corresponding initial boundary value problem with linear damping: $\ddot{u} - u_{xx} + (2/\pi)^2\dot{u} = 0$.

Because the initial data (52) is smooth then by Theorem 3.1 the unique solution u is, among other things, in $C^1_T C^1(\overline{\Omega})$. Consequently u must coincide with the solution to the ansatz (53).

In turn, (54) is a dissipative 2×2 system of ODEs and can be approximated by a Runge-Kutta scheme. To get some quantitative estimate on the absolute error of solutions found in Section 6.2, at least for initial data of the form (52), one can consider a piecewise linear interpolation of (54) and then calculate the energy-norm difference from the finite-element solution.

6.4 Energy plots

The accompanying figures and data demonstrate some of the numerical results. The initial data is considered of the form

$$u_0(x) = \frac{2}{\pi k}\sin(k\pi x), \quad u_1 \equiv 0$$

which permits to compare the finite element solutions to the pointwise Runge-Kutta solutions described in Section 6.3.

Figure 1 shows the point-value of displacement $u(t, 0.5)$ next to the displacement value at the same $x = 0.5$ for the corresponding initial boundary value problem *with linear damping.*

Figure 2 presents numerical estimates of the energy for solutions obtained by Ritz-Galerkin finite element scheme and successive approximations. The graphs indicate that the energy decay deteriorates as the frequency of the initial data goes up while the initial finite-energy remains fixed ($E_{u^{(k)}}(0) = 1$ independently of k), thus illustrating the lack of uniform which was rigorously confirmed by Theorem 3.2. The initial data are of the form (52) with zero initial velocity. The indicated errors are obtained by comparing each finite-element solution to a piecewise-linear interpolant of the corresponding piecewise RK solution (53).

Figure 3 uses multi-step extensions of the same solutions shown in Figure 2 to a larger time-scale using (5-step) Adams-Bashforth method. It also includes the decay of the $L^2(\Omega)$-norm $|u^{(k)}(t)|_0$ for these solutions.

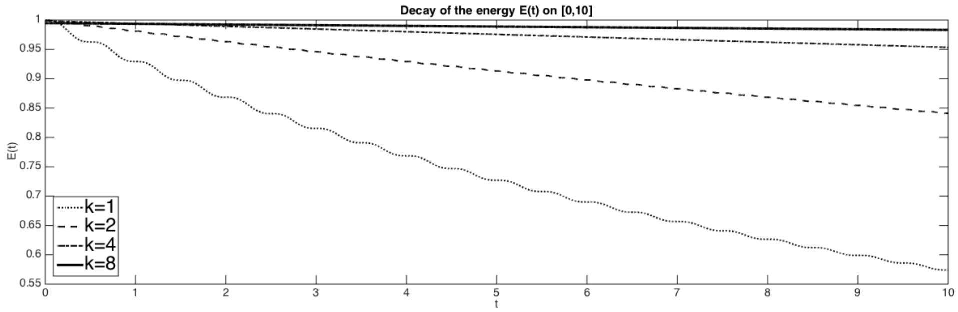

Figure 2: Plots of energy $E_{u^{(k)}}(t)$ versus time t for system (1)–(3) with $f(s) = s^2$. Time interval: $[0, T = 10]$. Obtained by approximating the Ritz-Galerkin semi-discrete solution using numerical integration in the variation of parameter formula. Element size: $h = 10^{-2}$; time-step: $\delta = 2 \cdot 10^{-3}$. Initial displacements: $u_0^{(k)} = 2(k\pi)^{-1}\sin(k\pi x)$ for frequencies $k = 1, 2, 4, 8$. Initial velocities are zero. Approximate error was obtained by computing maximal, over $[0, T]$, energy ($\mathscr{H}^0$-norm) difference from a piecewise-linear interpolation of the pointwise RK solution given by ansatz (53). The initial energy of each solution is 1. Corresponding errors e_k are: $e_1 = 3.84\text{E-}02$, $e_2 = 5.20\text{E-}03$, $e_4 = 6.97\text{E-}03$, $e_8 = 4.15\text{E-}01$.

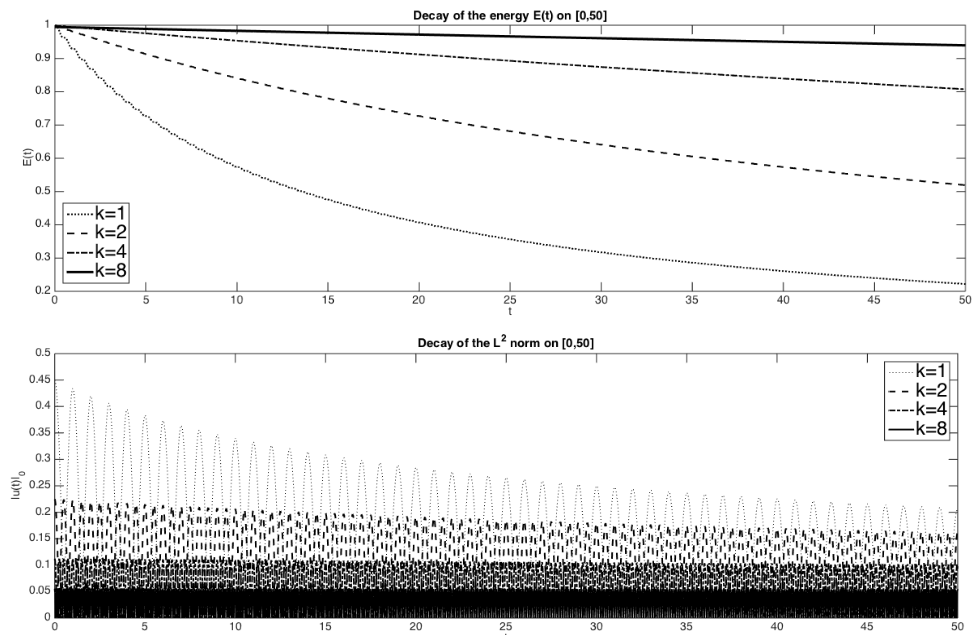

Figure 3: Plots of energy $E_{u^{(k)}}(t)$ and $L^2(\Omega)$ norm $|u^{(k)}(t)|_0$ vs time t for a numerical solution of problem (1)–(3) with $f(s) = s^2$. This is an extension of the solutions from Figure 2 (originally defined for $t \in [0, 10]$) to the time-interval $[0, 50]$ by 5-step Adams-Bashforth method. Due to rapid oscillations, the $L^2(\Omega)$ norm values for the solution corresponding to $k = 8$ appear to fill out a solid region.

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A Finite-dimensional counterpart

To complement the analysis of the infinite-dimensional model (1)–(3) it is also interesting to examine the related finite-dimensional version of a degenerately damped harmonic oscillator:

$$\ddot{x} + kx + f(x)\dot{x} = 0, \quad (k > 0) \tag{55}$$

$$x(0) = x_0, \quad \dot{x}(0) = x_1 \tag{56}$$

for $f = \alpha s^{2m}$, $m \in \mathbb{N}$. Rewrite it as a first order evolution problem

$$\dot{y} = G(y) \quad \text{for} \quad G(y) := \begin{bmatrix} y_2 \\ -f(y_1)y_2 - ky_1 \end{bmatrix}, \quad y(0) = \begin{bmatrix} x_0 \\ x_1 \end{bmatrix}. \tag{57}$$

Henceforth, let $|\cdot|$ denote an equivalent norm on $\mathbb{R}^2$:

$$|v|^2 := \tfrac{1}{2}k|v_1|^2 + \tfrac{1}{2}|v_2|^2 . \tag{58}$$

Because f is smooth and non-negative, then classical ODE results guarantee that solutions are unique, exist globally and satisfy

$$|y(t)| \le |y(0)| \quad \text{for all} \quad t \ge 0 .$$

Below we present stability results which contrast their infinite-dimensional analogues discussed earlier. In particular, the finite-dimensional system is uniformly stable, while for distributed-parameter version the strong stability is open while uniform stability has been proven false.

Lemma A.1. *The dynamical system corresponding to the ODE* (55)–(56) *is asymptotically ("strongly") stable.*

Proof. This is a direct consequence of LaSalle's invariance principle with the Lyapunov function given by the equivalent norm (58): $V(y) = |y|^2$. We only need to check what kinds of trajectories reside in the invariant set of the system:

$$E = \{y \in \mathbb{R}^2 : \nabla V(y) \cdot G(y) = 0\} .$$

So for $y \in E$

$$ky_1y_2 - f(y_1)y_2^2 - ky_1y_2 = 0 .$$
$$f(y_1)y_2^2 = 0$$

In particular, either $y_1 = 0$ or $y_2 = 0$.

Now suppose that a solution trajectory $\{(t, y = (x(t), \dot{x}(t))) : t \geq 0\}$ resides in E. If at some t we have $x(t) \neq 0$, then by the continuity in time, it is nonzero on some interval I. On that interval we must have $\dot{x} \equiv 0$ by the property of E. But then on that interval, from equation (55) we get a contradiction since the solution has to be constant and therefore zero.

Thus the only trajectory in E is the trivial one. By LaSalle's invariance principle every solution is asymptotically stable. $\square$

Theorem A.1. *The dynamical system corresponding to the ODE* (55)–(56) *is uniformly stable.*

Proof. Proceed by contradiction. Assume for a bounded set $B \subset \mathbb{R}^2$ there exists some $\varepsilon > 0$, a bounded sequence of initial data $(y_{0n})_n \subset B$, and a sequence of corresponding times $(T_n)_n$ with $T_n \nearrow \infty$ such that

$$|y_n(T_n)| > \varepsilon\,.$$

Extract a convergent subsequence of initial data, reindexed again by n. Let $z_0 \in \overline{B}$ denote this limit point and $t \mapsto z(t)$ be the corresponding solution. By Lemma A.1

$$\lim_{t\to\infty} z(t) = \mathbf{0}\,.$$

In particular, there exists T such that for $t \geq T$ we have $|z(t)| < \varepsilon/2$. Because the non-linearity $(x_1, x_2) \mapsto f(x_1)x_2$ is locally Lipschitz on $\mathbb{R}^2$, and the system (57) is non-accretive, then there exists $\delta = \delta(T, \overline{B}) > 0$ such that for any $\eta_0 \in B$ if $|\eta - z_0| < \delta$, the corresponding solutions satisfy

$$|\eta(t) - z(t)| < \frac{\varepsilon}{2} \quad \text{for} \quad t \in [0, T]\,.$$

Since $T_n \nearrow \infty$, we can find $T_N > T$. Next, because y_{0n} converge to z_0, then for $n > N$ we can find find y_{0n} so that $|y_{0n} - z_0| < \delta$ and, consequently,

$$|y_n(t) - z(t)| < \frac{\varepsilon}{2} \quad \text{for} \quad t \in [0, T]\,.$$

Then $|y_n(T)| < \varepsilon/2 + |z(T)| < \varepsilon$. Because the system is non-accretive, then $|y_n(t)| < \varepsilon$ for all $t \geq T$, and in particular it holds for $T_n \geq T_N > T$. So $|y_n(T_n)| < \varepsilon$ which contradicts the choice of y_{0n}. $\square$

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